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13. ABSTRACT *(Maximum 200 words)*

Specific aims of our projects have been to study the following topics:
(i) Backward Stochastic Differential Equations with reflection and connection with Dynkin games;
(ii) A deterministic approach to discrete-time Dynkin games;
(iii) Singular control problems with application to irreversible investment;
(iv) Synchronization and optimality for d-armed bandit problems;
(v) Adaptive control of a diffusion to a goal, and connection to a parabolic Monge-Ampere-type equation;
(vi) Backward Stochastic Differential Equations with Constraints on the gains-process;
(vii) Control and stopping of a diffusion process on an interval;
Findings include establishing existence and uniqueness results for solutions to these problems and their analytical characterizations. These results are significant because they provide explicit ways of constructing solutions or even explicit expressions for optimal processes in the above problems.

14. SUBJECT TERMS
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FINAL PROGRESS REPORT

STATEMENT OF THE PROBLEM STUDIED AND SUMMARY OF THE MOST IMPORTANT RESULTS

In article [1] we study the topic of Backward Stochastic Differential Equations (BSDEs) and establish existence and uniqueness results for solutions to such equations in the presence of two reflecting barriers, generalizing previous works on BSDEs, in particular El Karoui et al. (1997). We also demonstrate that the solution coincides with the value of a certain Dynkin game, a stochastic game of optimal stopping. The game involves two players, each of whom can decide to stop it at a random time of his choice; upon termination a certain amount is paid by one of the players to the other. Moreover, the connection with the BSDE enables us to provide a deterministic approach to the game. One can also think of the solution as a "guided missile" that has to hit a target at some future time T, but is also required to remain inside a certain (moving and random) region before that time.

The discrete-time version of these results, including both the constrained Dynamic Programming Equation (analogue of the BSDE) and the pathwise approach to the discrete-time stochastic game, are developed in paper [2].

Article [3] studies the *Monotone Follower Problem in Singular Stochastic Control* and its connections with Optimal Stopping. These connections are developed in the context of the "irreversible investment" problem in mathematical economics, first considered by K. Arrow, and allow "elementary" treatment of this problem in complete generality.

In article [4] we completed our definitive treatment of the general multi-armed bandit (dynamic allocation) problem in continuous time, for arms that are not necessarily independent or Markovian. This work complements our previous major contribution to this subject, which provides a similar treatment for the discrete-time case (El Karoui & Karatzas (1993)). Notions and results from time-changes, optimal stopping theory, and multi-parameter martingale theory were used extensively in this treatment, and the traditional independence assumption was replaced by the famous condition (F.4) of Cairoli & Walsh, which allows for some dependence in the evolution of the various arms (or projects). In this work we also introduced a "synchronisation identity" for allocation strategies, and brought out its central role in the context of dynamic allocation: this condition is both necessary and sufficient for optimality in the case of decreasing rewards, and leads to an allocation strategy with all the important properties in the general case as well, namely

(i) optimality in the dynamic allocation problem,

(ii) optimality in a dual (minimization) problem, and

(iii) the "index-type" property of Gittins.

Even when specialized to the case of "diffusion" or "Brownian" Bandits, this synchronisation propery leads to deep results, as we demonstrate in the paper by examples. We believe that this is a seminal study of the problem, and that it will lead to renewed interest in this and similar questions.

In work [5], we solve the following adaptive control problem: to maximize the probability $P[X(T) = 1]$, subject to the dynamics $dX(t) = u(t)[Bdt + dW(t)], X(0) = x, 0 < x < 1$, where

(i) B is a random variable with known distribution, independent of the Brownian motion $W(\cdot)$,

(ii) $u(\cdot)$ is a real-valued control process, square-integrable over the interval $[0, T]$ with probability one, and adapted to the filtration generated by the Brownian-motion-with-unknown-drift process $Y(t) = W(t) + Bt$, and

(iii) the state-process $X(\cdot)$ takes values in the interval $[0, 1]$ and is absorbed when it reaches either one of the endpoints of the interval.

We obtain very explicit expressions for both the optimal control, and for the value-function of this Bayesian adaptive control problem. In the process, we also obtain explicit solutions to the parabolic Monge-Ampère-type equation

$$2Q_{yy}Q_s = Q_{xx}Q_{yy} - (Q_{xy})^2$$

which is related to the HJB equation of this problem.

In article [6] we develop a general theory of Backward Stochastic Differential Equations with convex constraints on the gains (or intensity-of-noise) process. We establish existence and uniqueness of a minimal solution under Lipschitz and convexity assumptions on the drift coefficient. We also show that the minimal solution can be characterized as the unique solution of a functional stochastic-control-type equation. This representation is important because it provides an explicit scheme – a penalization method – for constructing the minimal solution. Moreover, we provide a similar analysis in the more general case in which there is also a reflecting lower-barrier for the state-process.

The paper [7] discusses the problem of optimal stopping for a controlled diffusion process on the unit-interval $I = [0, 1]$ with absorbing barriers, and with drift/diffusion pair that can be selected at any instant $t \in [0, \infty)$ from a set depending only on the current position. We reduce this mixed optimal stopping/control problem effectively to two optimal stopping problems, and describe explicitly its optimal policies.

In technical report [8] we generalize our previous results [1] on existence and uniqueness of solutions to one-dimensional BSDEs to the multi-dimensional and Forward-Backward case, in which the "backward" process has to hit the "forward" process at time T, and they are both restricted to remain in a certain region. In the case of a Markovian, one-dimensional model, we characterize the solution of the Forward-Backward SDE as the unique viscosity solution to a quasilinear variational inequality (obstacle problem) with a Neumann boundary condition.

LIST OF PUBLICATIONS AND TECHNICAL REPORTS

[1] J. CVITANIC & I. KARATZAS (1996) Backward Stochastic Differential Equations with reflection and Dynkin games. *Annals of Applied Probability* **24**, 2024-2056.

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[8] J. CVITANIC & J. MA (1996) Reflected Forward-Backward SDEs and obstacle problems with boundary conditions. Technical Report.

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